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On the convergence of certain sequences of rational approximants to meromorphic functions in several variables☆

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Abstract

In a previous paper, the author introduced a new class of multivariate rational interpolants, which are called Optimal Padé-type Approximants (OPTA). There, for this class of rational interpolants, which extends classical univariate Padé Approximants, a direct extension of the "de Montessus de Ballore's Theorem" for meromorphic functions in several variables is established. In the univariate case, this theorem ensures uniform convergence of a row of Pade Approximants when the denominator degree equals the number of poles (counting multiplicities) in a certain disc. When one overshoots the number of poles when fixing the denominator degree, convergence in measure or capacity has been proved and, under certain additional restrictions, the uniform convergence of a subsequence of the row. The author tackles the latter case and studies its generalization to functions in several variables by using OPTA.

Keywords: Multivariate Padé Approximants; Convergence of approximants; Meromorphic functions in several variables

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1. Introduction

The subject of this paper is the extension of de Montessus de Ballore-type theorems to the multivariate case. We start with a brief overview on the current state of the problem. First, in the univariate case, it is well known that the classical de Montessus de Ballore's Theorem asserts the uniform convergence of the sequence of $\{[n/m]\}_{n\in\mathbb{N}}$ Padé Approximants (in the sequel PA) to a function *f*, being meromorphic in a certain disk $D = D(0, R)$ with precisely m poles in D (counting multiplicities), in compact subsets of $D\setminus\{z_1, \ldots, z_m\}$, where $z_1, \ldots, z_m$ are the poles of *f* in *D*. Moreover, we know that each pole of *f* attracts as many poles of PA as its multiplicity. It is clear that this result deals with the problem of describing the *m*-meromorphic extension of an analytic function in a neighborhood of the origin in terms of the asymptotic distribution of the poles of PA (see e.g. [6]). Also, in this sense, one can consider the problem of the convergence of $\{[n/m]\}_{n\in\mathbb{N}}$ to *f* in its disk of *m*-meromorphy; that is, the maximal disk $D_m = D(0, R_m)$ where *f* has at most *m* poles counting multiplicities. Let μ the number of poles of *f* in D_m. If *f* possesses precisely *m* poles in D_m, i.e. $\mu = m$, the classical de Montessus de Ballore's Theorem works and if $\mu = m - 1$, Buslaev et al. show, by applying certain results due to Hadamard, the uniform convergence of a subsequence of $\{[n/m]\}_{n\in\mathbb{N}}$ to *f* in compact subsets of $D\setminus\left\{z_1, \ldots, z_\mu\right\}$ (see [7]). For the general case $0\leqslant\mu\leqslant m$, Baker and Graves-Morris conjectured (see e.g. [4]) that the same conclusion was valid. However, in [7] this conjecture was rejected by means of a simple counterexample for the case $m = 2$ and $\mu = 0$. In this sense, a result of general character is contained in the following theorem (see e.g. [22, p. 90] or [7, p. 539]).

Theorem 1.1. *Let f be holomorphic in a neighborhood of the origin and let* $\mu < m$ *the number of poles of f in* $D_m = D(0, R_m)$, *with* $m > 0$ *a non-negative integer. If the poles of f are denoted by* $\left\{z_1, \ldots, z_\mu\right\}$, *then there exists a subsequence of* $\{[n/m]\}_{n\in\mathbb{N}}$ *converging uniformly* (*even geometrically*) *to f in compact subsets of* $D_m\setminus\left(\left\{z_1, \ldots, z_\mu\right\}\cup S\right)$, *where the set S contains a number of points less than or equal to* $m - \mu - 1$. *Moreover*, *each pole of f attracts as many poles of* $[n/m]$ *as its multiplicity.*

Under additional restrictions, another result of general character for the convergence of subsequences was given in [3].

Another approach which extends de Montessus de Ballore's Theorem is related to the use of weaker versions of convergence, such as convergence in capacity or in Hausdorff measure. In this sense, it is possible to give results of convergence of the whole sequence $\{[n/m]\}_{n\in\mathbb{N}}$. Moreover, convergence in capacity can be achieved even for sequences of the type $\{[n/m_n]\}_{n\in\mathbb{N}}$, where $\liminf\, m_n\geqslant\mu$ and $\lim\,(m_n/n) = 0$ (see [19]).

There is a considerable amount of difficulty in the extension of this theory to the multivariate case, which is the purpose of the present paper. Thus, the direct extension of de Montessus de Ballore's Theorem to several variables, with rational approximants determined by the "accuracy-through-order" principle (see [12]), is a problem which, in a general sense at least, can not be solved (see the counterexamples given in [22]), even though several approaches exist in this direction ([10,23,13,14]). Nevertheless, several authors observed problems in the proofs in these works (see e.g. [22, p. 95] or [18, p. 213]). Moreover, approaches made following principles other than "accuracy-through-order" (by Chaffy [8], Cuyt [12,17], and

Guillaume [20]) do not provide a totally general extension of de Montessus de Ballore's Theorem in the following sense: if we have a meromorphic function in a polydisc $P(0, R)$ and Q is a minimal polynomial which kills the poles of f in the polydisc, then, in general, none of these approaches guarantee uniform convergence of the respective rational approximants in the whole polydisc. To this end in [18] we introduced a new class of multivariate rational approximants, which we call OPTA, that is, *Optimal Padé-type Approximants* (in fact, they are Padé-type Approximants, using the terminology due to Brezinski [5] to design the rational interpolants with prescribed denominators, see also [2] or [1] for the multivariate case), in which the usual "accuracy-through-order" principle to determine the denominator is replaced by certain minimal norm conditions. A similar approach was independently followed by Guillaume et al. [21], in such a way that their approximants may be seen as a particular case of our OPTA. For this new class of rational approximants, which extends the classical univariate PA, we proved in [18] two theorems which provide the extension of de Montessus de Ballore's Theorem for sequences of $\{[N_k/M]\}_{k\in\mathbb{N}}$ OPTA of f (N_k and M denote the respective exponent sets for the numerators and denominators of the OPTA). In this case, there exist a complete Reinhardt domain $\mathfrak{D}$ (for the definition see e.g. [23, pp. 32–33]) and a non-zero polynomial Q with exponent set M, uniquely determined up to a multiplicative constant, such that $\mathfrak{D}$ is the domain of the power series of Qf. But, what can be said when M is "larger" than necessary; that is, when Q is not unique? The present paper is essentially devoted to give an answer to this question and provide a multivariate counterpart of Theorem 1.1. In this sense, we must point out that there exist similar results due to Cuyt and Lubinsky, but only for the *Multivariate Homogeneous Padé Approximants* (see [12,17]).

On the other hand, I wish to point out that Montessus-type theorems using convergence in measure or capacity have been given for multivariate functions in [15], following the "accuracy-through-order" approach, and in [16], for the homogeneous approach. We shall deal with such extensions for our OPTA in a forthcoming paper.

The paper is organized as follows. In Section 2, we summarize the definition and some algebraic properties contained in our previous paper [18], while in Section 3 the convergence results of this article are stated. In Section 4 these results and the computational viability of these approximants are illustrated by means of some numerical examples. Finally, in Section 5 the proofs of the main results are shown.

2. Auxiliary results

Hereafter we make use of standard multi-index notation, that is, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, $v = (v_1, \ldots, v_d) \in (\mathbb{R}\setminus\{0\})^d$, and $\lambda \geqslant 0$ we denote:

$\alpha! = \alpha_1! \cdots \alpha_d!$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\Pi_j(z) = z_j$, $\Pi_j(\alpha) = \alpha_j$, $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, and $\lambda^v = \left(\lambda^{v_1}, \ldots, \lambda^{v_d}\right)$. Furthermore, for any $z, z' \in \mathbb{C}^d$, we will write $< z', z >= \sum_{i=1}^d z_i' \overline{z_i}$ and $zz' = \left(z_1 z_1', \ldots, z_d z_d'\right)$.

In the same way, for two given sets $A, B \subset \mathbb{N}^d$, the "sum" of these sets (related to the set of exponents corresponding to the product of two polynomials) is defined by

$$A + B = \{(a + b) : a \in A,\ b \in B\}.$$

Analogously, the "difference" set is given by

$$A - B = \{(a - b) : a \in A,\ b \in B\} \cap \mathbb{N}^d.$$

Now, we proceed to recall the definition and main properties of the new class of multivariate rational interpolants introduced in [18]. We start with a definition concerning linear mappings.

Definition 2.1. Let $T : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be a linear mapping with $m \in \mathbb{N}\setminus\{0\}$ and $n \in \mathbb{N}$ and let δ be a real number such that $\delta \geqslant 1$. Then, $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$ is said to be a *strong pseudominimum of T* for $[m, n, \delta]$ with respect to a certain norm $\|.\|$ in $\mathbb{C}^n$ if $x_1 = 1$ and

$$\|Tx\| \leqslant \delta \min_{y_1=1} \|Ty\|.$$

In a similar way, we say that $x \in \mathbb{C}^m$ is a *weak pseudominimum of T* for $[m, n, \delta]$ with respect to the norms $\|.\|$ in $\mathbb{C}^n$and $\|.\|_*$ in $\mathbb{C}^m$ if $\|x\|_* = 1$ and

$$\|Tx\| \leqslant \delta \min_{\|y\|_*=1} \|Ty\|.$$

Making use of the definition above, our new class of multivariate Padé-type Approximants, which we call OPTA, is introduced as follows. Let $d \in \mathbb{N}\setminus\{0\}$ and consider a (possibly formal) power series $f(x) = \sum_{\alpha\in\mathbb{N}^d} f_\alpha x^\alpha$ (if $\alpha \notin \mathbb{N}^d$, we define $f_\alpha = 0$ for consistency).

Definition 2.2. If N, M are two finite subsets in $\mathbb{N}^d$ with $0 \in M$, R is a polyradius $R > 0$ (hereafter it means that $R = (R_1, \ldots, R_d)$ with $R_i > 0$, for $i = 1, \ldots, d$) and $\delta \geqslant 1$, we say that the rational function r is a strong OPTA of f for $[N, M, R, \delta]$ if the following holds:

(a) $r = \frac{p}{q}$, with $p \in \pi_N$, $q \in \pi_M$ (if L is a finite subset of $\mathbb{N}^d$ and t is a polynomial, hereafter the notation $t \in \pi_L$ means that L is the exponent set of t).
(b) Considering the set $E = E(N, M) = ((N + M) - M)\setminus N$ and setting $q(x) = \sum_{\beta\in M} q_\beta x^\beta$ and the linear function $T : \mathbb{C}^{\#M} \longrightarrow \mathbb{C}^{\#E}$, such that for $M \neq \{0\}$ maps the vector $u = (u_\beta)_{\beta\in M}$ onto the vector $v = \left(\sum_{\beta\in M} u_\beta f_{\alpha-\beta} R^\alpha\right)_{\alpha\in E}$, then the vector $(q_\beta)_{\beta\in M}$is a strong pseudominimum of T for $[\#M, \#E, \delta]$ with respect to the norm $\|.\|_1$ in $\mathbb{C}^{\#E}$, where now we take u_0 as the first component of the vector $u \in \mathbb{C}^{\#M}$.
(c) p is the Taylor polynomial of the function fq with N as its exponent set; that is, $(fq - p)(x) = \sum_{\alpha\in\mathbb{N}^d\setminus N} e_\alpha x^\alpha$.

Remark 2.3. Under the same conditions, we say that r is a weak OPTA of f for $[N, M, R, \delta]$ when the requirements above are satisfied, but now in (b) the vector $(q_\beta)_{\beta\in M}$ is taken as a weak pseudominimum of T for $[\#M, \#E, \delta]$ with respect to the norm $\|.\|_1$ in $\mathbb{C}^{\#E}$ and the norm $\|.\|_\infty$ in $\mathbb{C}^{\#M}$ (Definition 2.2 in [18]).

Remark 2.4. As we proved in [18, Proposition 2.3] this class of strong (weak) OPTA extends the classical univariate strong (respect. weak) Padé Approximants, where if there is no interpolation defect in the rational interpolation problem, the solution is said to exist in a strong sense (Baker's definition of PA in [4]), while if there is an interpolation defect and only the linear version of the rational interpolation problem has a solution, then this solution is said to be a weak solution (Padé–Frobenius' definition of PA in [4]).

Since our aim in the present paper is to provide results of geometrical convergence of OPTA sequences, we now state the following definition:

Definition 2.5. Let *f* and *R* be as above, $(N_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$ are two sequences of finite subsets in $\mathbb{N}^d$ with $0 \in M_k$ for each *k*, and $\sigma = (\sigma(k))_{k\in\mathbb{N}}$ and $\delta = \left(\delta_k\right)_{k\in\mathbb{N}}$ two sequences of real numbers in $(0, \infty)$ and $[1, \infty)$, respectively, such that $\lim_{k\to\infty} \sigma(k) = \infty$ and $\lim_{k\to\infty} \left(\delta_k\right)^{1/\sigma(k)} = 1$. A sequence of rational functions $(r_k)_{k\in\mathbb{N}}$ is said to be a *σ-geometrically strong* (*weak*) *OPTA of f for* $\left[(N_k)_{k\in\mathbb{N}}, (M_k)_{k\in\mathbb{N}}, R, \delta, \sigma\right]$ if for each $k \in \mathbb{N}$, r_k is a strong (weak) OPTA of *f* for $\left[N_k, M_k, R, \delta_k\right]$.

On the other hand, since from the definition of OPTA the computational viability of these approximants does not seem clear, we must point out that the definition above does not essentially depend on the norm, which enables us to replace the ℓ_1-norm by any ℓ_p-norm (for instance, $p = 2$). Indeed, in practice (see the numerical examples displayed in Section 4), these OPTA can be computed by a straightforward procedure, since their denominators arise as least-squares solutions of overdetermined systems of linear equations.

3. Convergence results

In order to establish our main theorems we need a previous result concerning some algebraic aspects in the theory of functions of several variables. As far as we know, there is no proof of such a result in the literature, for which we include a complete proof of it. Moreover, we think that it is of independent interest.

Let Ω be an open set in $\mathbb{C}^d$ and *f* a holomorphic function in the open set $\Theta \subset \Omega$. For the pair (f, Ω), denote $\mathrm{I} = \mathrm{I}(f, \Omega) = \{p \text{ polynomial: } (fp) \in \mathcal{O}(\Omega)\}$, where as usual, the notation $g \in \mathcal{O}(D)$ means that the function *g* (or some extension of it) is holomorphic in some open set containing *D*. It is clear that I is an ideal of $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_d]$, the set of polynomials in *d* variables with complex coefficients. Under these conditions, we have

Proposition 3.1. *The set* I *is a principal ideal*; *i.e.*, *there exists a polynomial p in* $\mathbb{C}[x]$ *so that* $\mathrm{I} = (p)$. *That is,* I *is generated by p.*

Now we are in a position to state the extensions of Theorem 1.1 to the multivariate case. First, let us specify some notations.

Indeed, let $(N_k)_{k\in\mathbb{N}}, (M_k)_{k\in\mathbb{N}}$ and $(E_k)_{k\in\mathbb{N}} \subset \mathbb{N}^d$ be as above, where $E_k = E(N_k, M_k)$. For any vector $v \in \mathbb{R}^d$ with $v > 0$, denote by $\Lambda_v(k) = \min\left\{< v, \alpha >: \alpha \in \mathbb{N}^d \backslash N_k\right\}$ and $\sigma_v(k) = \min\{< v, \alpha >: \alpha \in E_k\}$ (if $E_k = \varnothing$ then $\sigma_v(k) = \Lambda_v(k)$), where for simplicity we write $\Lambda_1(k) = \Lambda_{(1,\dots,1)}(k)$ and $\sigma_1(k) = \sigma_{(1,\dots,1)}(k)$. Hereafter, for a complete Reinhardt domain $\mathfrak{D}$ in $\mathbb{C}^d$, a vector $v \in \mathbb{R}^d$ with $v > 0$, and a polyradius $R > 0$, we shall denote $\rho_v(R, \mathfrak{D}) = \inf\{\lambda > 0 : P(0, R\lambda^{-v}) \subset \mathfrak{D}\}$, where $P(z, r)$ denotes the polydisc centered in $z \in \mathbb{C}^d$ and with polyradius $r > 0$ (observe that if $\overline{P(0, R)} \subset \mathfrak{D}$, then $\rho_v(R, \mathfrak{D}) < 1$).

In what follows, we suppose that f is a holomorphic function in a neighborhood of the origin in $\mathbb{C}^d$, $Q(x) = \sum_{\beta \in M} Q_\beta x^\beta$ is a polynomial, with exponent set $M \subset \mathbb{N}^d$, $0 \in M$ and $Q(0) > 0$ ($Q_{\beta_0} > 0$, for some $\beta_0 \in M$, respectively), and we denote by $\mathfrak{D}$ the domain where the Taylor expansion of Qf converges.

Theorem 3.2. *Take a polyradius $R > 0$ such that $\overline{P(0, R)} \subset \mathfrak{D}$. Let $(N_k)_{k=1}^\infty$ be a sequence of finite sets in $\mathbb{N}^d$ such that $\lim_{k\to\infty} \Lambda_1(k) = \infty$ and $(r_k)_{k=1}^\infty$ be a σ_1-geometrically strong (respect. weak) OPTA of f for $\left[(N_k)_{k=1}^\infty, (M)_{k=1}^\infty, \delta, R, \sigma_1\right]$, where δ is taken so that $\lim_{k\to\infty} (\delta_k)^{1/\sigma_1(k)} = 1$. For this sequence and for any $k \in \mathbb{N}$, consider $r_k = \frac{\tilde{p}_k}{\tilde{q}_k}$, where $\tilde{q}_k$ and $\tilde{p}_k$ denote the normalization of the polynomials q_k and p_k in order to satisfy that $1 = \max_{\beta\in M} |\tilde{q}_{\beta,k}|$.*

Then, for each subsequence $(\tilde{q}_{k_j})_{j=0}^\infty$ converging to a polynomial $P(x) = \sum_{\beta\in M} P_\beta x^\beta$, with $1 = \max_{\beta\in M} |P_\beta|$, (in fact, such a subsequence always exists) we have that $Pf \in \mathcal{O}(P(0, R))$.

Moreover, for each $v \in \mathbb{R}^d$ with $v > 0$, $\mu \in [0, 1]$ and $\varepsilon > 0$, if we denote $r = R\mu^v$ and $L_\varepsilon = \{x \in \mathbb{C}^d : |QP(x)| < \varepsilon\}$, we have

$$\varlimsup_{j\to\infty} \left(\| f - r_{k_j} \|_{\infty, \overline{P(0,r)}\setminus L_\varepsilon}\right)^{1/\Lambda_v(k_j)} \leqslant \rho_v(r, \mathfrak{D}) < 1. \tag{3.1}$$

Remark 3.3. Observe that if $(p) = I(f, P(0, R))$, then p/P; that is p divides to P. In the special case where $Q = p$, we have Q/P. Moreover, it is easy to see that the conclusion of Theorem 3.2 above holds for the whole sequence of strong (respect. weak) OPTA, where now $\tilde{q}_k$ converges to Q, normalizing $\tilde{q}_k$ and Q so that $\max_{\beta\in M} |\tilde{q}_{\beta,k}| = \max_{\beta\in M} |Q_\beta| = 1$, $\tilde{q}_k(0) \geqslant 0$ and $Q(0) > 0$ (respect. $\tilde{q}_{\beta_0,k} \geqslant 0$ and $Q_{\beta_0} > 0$, for some $\beta_0 \in M$), provided that Q is M-maximal (definition introduced in [21]); that is:

$$\text{if } P \in \pi_M \text{ and } Q/P, \text{ then } P = cQ \text{ with } c \in \mathbb{C}.$$

Finally, if the conditions of Theorem 2.4 in [18] hold, it is easy to see that Q is M-maximal and that the conclusion above is valid, but Theorem 2.4 in [18] also ensures the geometrical convergence of the sequence $\tilde{q}_k$ to Q.

Theorem 3.4. *Let $R > 0$ an arbitrary polyradius. For $u \in \mathbb{R}^d$ with $u > 0$, denote $S_u(k) = \max\{< u, \alpha >: \alpha \in E_k\}$ (if $E_k = \emptyset$ we take $S_u(k) = \sigma_u(k)$) and suppose that $\lim_{k\to\infty} \frac{S_u(k)}{\sigma_u(k)} = 1$ and that, without loss of generality, $(Q) = \mathrm{I}\left(f, P\left(0, \tilde{R}\right)\right)$, where $\tilde{R} = \sup\{R\lambda^u : \lambda > 0 \text{ and } P(0, R\lambda^u) \subset \mathfrak{D}\}$. For each k, consider $r_k = \frac{\tilde{p}_k}{\tilde{q}_k}$, with $\tilde{q}_k$ and $\tilde{p}_k$ as in Theorem 3.2.*

Then, for each subsequence $\left(\tilde{q}_{k_j}\right)_{j=0}^{\infty}$ *converging to a polynomial* $P(x) = \sum_{\beta \in M} P_\beta x^\beta$, *with* $1 = \max_{\beta \in M} \left|P_\beta\right|$, (*in fact, such a subsequence always exists*) *we have that* Q/P.

Moreover, in both strong and weak cases, for each $\lambda > 0$ *so that* $P\overline{\left(0, R\lambda^u\right)} \subset \mathfrak{D}$, *and for each* $\varepsilon > 0$, $\mu \in [0, 1]$ *and* $v \in \mathbb{R}^d$ *with* $v > 0$, *if we denote* $r = R\lambda^u \mu^v$ *and* $L_\varepsilon = \left\{x \in \mathbb{C}^d : |P(x)| < \varepsilon\right\}$, *we have*

$$\overline{\lim_{j\to\infty}} \left(\left\| f - r_{k_j} \right\|_{\infty, P\overline{(0,r)} \setminus L_\varepsilon}\right)^{1/\Lambda_v(k_j)} \leqslant \rho_v\left(r, \mathfrak{D}\right) < 1. \tag{3.2}$$

Remark 3.5. As in [18], we give two slightly different extensions to Theorem 1.1. In Theorem 3.2, the numerator lattices $(N_k)_{k=1}^{\infty}$ can be chosen with total freedom, but we need to select a suitable polyradius R to ensure the convergence of OPTA. On the contrary, in Theorem 3.4 if the sequence of numerator lattices satisfies certain natural condition, the results on convergence are valid in a larger set, independent of the choice of the polyradius. To illustrate this difference, consider the univariate case. In fact, in the particular case when $d = 1$ and $M = \{0, 1, \ldots, m\}$, with $m = \#M - 1$, in order to apply Theorem 3.2 we have total freedom to select the sequence $(N_k)_{k=1}^{\infty}$, but we must choose a radius R belonging to the interval $(0, R_m)$, where for each $n \in \mathbb{N}$, R_n denotes the n-meromorphy radius of f. In this situation, the convergence is achieved in compact subsets of $P\overline{(0, R)} \setminus P^{-1}(\{0\})$. On the contrary, if the natural condition $\lim_{k\to\infty} \dfrac{S_1(k)}{\sigma_1(k)} = 1$ is satisfied, then by applying Theorem 3.4, we can guarantee convergence in compact subsets of the larger set $P(0, R_m) \setminus P^{-1}(\{0\})$, for any radius $R > 0$. Moreover, if $(Q) = \mathrm{I}(f, P(0, R_m))$ with Q a polynomial of degree $\mu \leqslant m$, then in order to conclude that Q/P, in Theorem 3.2 we must choose a radius belonging to the interval $\left(R_{\mu-1}, R_m\right)$, while in Theorem 3.4 we can choose any radius $R > 0$. In this sense, we consider Theorem 3.4 as the proper extension of the univariate Theorem 1.1, even when dealing with the univariate case, where the numerator lattices $(N_k)_{k=1}^{\infty}$ can be quite freely chosen but the set S contains a number of points less than or equal to $m - \mu$.

Remark 3.6. It is easy to see that the conclusion of Theorem 3.4 above holds for the whole sequence of strong (respect. weak) OPTA, where now $\tilde{q}_k$ converges to Q, normalizing $\tilde{q}_k$ and Q so that $\max_{\beta \in M} \left|\tilde{q}_{\beta,k}\right| = \max_{\beta \in M} \left|Q_\beta\right| = 1$, $\tilde{q}_k(0) \geqslant 0$ and $Q(0) > 0$ (respect. $\tilde{q}_{\beta_0,k} \geqslant 0$ and $Q_{\beta_0} > 0$, for some $\beta_0 \in M$), provided that Q is M-maximal. Finally, if the conditions of Theorem 2.5 in [18] hold, it is easy to see that Q is M-maximal and that the conclusion above is valid, but Theorem 2.5 in [18] also ensures the geometrical convergence of the sequence $\tilde{q}_k$ to Q.

4. Numerical examples

We now test the results on convergence of OPTA sequences to meromorphic functions analyzed in the previous section by means of some illustrative numerical examples. The results displayed in the tables below are related to the function $f(x, y) = \frac{\exp(x+y)}{1-2(x+y)+x^2+y^2}$,

Table 1

n	$E_{n,2;1}(z_1)$	$E_{n,3;1}(z_1)$	$E_{n,4;1}(z_1)$	$E_n(f)(z_1)$
3	.2601E-03	.1116E-04	.4348E-06	.2707E+00
6	.6023E-07	.2869E-09	−.2571E-12	.3388E-01
9	.2161E-11	.3109E-14	.0000E+00	.4235E-02
12	.0000E+00	.0000E+00	.4441E-15	.5294E-03
15	.0000E+00	.4441E-15	.0000E+00	.6618E-04
16	−.4441E-15	.0000E+00	.0000E+00	.3309E-04

with *fQ* being holomorphic in $\mathbb{C}^2$ when we take $Q(x, y) = 1-2(x+y)+x^2+y^2$. Numerical results of rational interpolation for this function have been previously shown in [13,18]. For each $n \in \mathbb{N}$ consider the sets $N_n = M_n = \left\{\alpha \in \mathbb{N}^2 : \alpha_1 + \alpha_2 \leqslant n\right\}$, and for $n, m \in \mathbb{N}$ and $s \in (0, \infty)$ denote by $r_{n,m;s}$ the unique (in this case) rational function for which the conditions of Definition 2.2 hold, with $N = N_n$, $M = M_m$ and $R = (s, s)$, so that in this case the denominator vector in Definition 2.2 (b) is taken as a strong pseudominimum of T for $[\#M, \#E, 1]$ with respect to the least-squares norm $\|.\|_2$ in $\mathbb{C}^{\#E}$. From Remark 2.9 in [18], $\left(r_{n,m;s}\right)_{n\in\mathbb{N}}$ is a σ_1-geometrically strong OPTA of *f* for $\left[(N_n)_{n\in\mathbb{N}}, (M_m)_{n\in\mathbb{N}}, R, \delta', \sigma_1\right]$, with σ_1 as in Theorem 3.2 and δ' as in Proposition 2.1 in [18]. It is easy to check that for $m \geqslant 2$ the hypotheses in Theorem 3.4 with $u = (1, 1)$ are fulfilled. These choices for the sets *N*, *M* and *R* are the most natural if we take into account the symmetry properties of *f*. Under these conditions, in the tables the error $\left(f - r_{n,m;s}\right)$ attained in a certain point $z \in \mathbb{C}^2$ is denoted by $E_{n,m;s}(z)$. All the calculations were performed with Microsoft Fortran Power Station.

Results displayed in Table 1 correspond to the point $z_1 = \left(\frac{1-\frac{1}{\sqrt{2}}}{2}, \frac{1-\frac{1}{\sqrt{2}}}{2}\right)$ which belongs to the domain of convergence of the Taylor series of *f*. It is easy to see that the speed of convergence is similar for the OPTA sequences corresponding to $m = 2, 3, 4$, although the first one seems to be, in principle, the most suitable. In addition, the speed of convergence of these three sequences is much faster than the corresponding for the sequence $(T_n(f)(z_1))_{n\in\mathbb{N}}$, where $T_n(f)$ denotes the nth Taylor polynomial for f; that is $(f - T_n(f))(x, y) = \sum_{(i,j)\in\mathbb{N}^2\setminus N_n} e_{i,j} x^i y^j$. The errors $E_n(f) = f - T_n(f)$ are displayed in the fifth column.

In Tables 2 and 3, we show the sequences of errors $E_{n,m;s}$ for the points $z_2 = (1, 1)$ and $z_3 = (1.64, 1.64)$ which are placed outside the domain of convergence of the Taylor series of *f* (in the first case we take $s = 4$ in addition to $s = 1$ to show that the results are rather independent of the choice of the polyradius). Again, we note that the convergence holds for the sequences (in spite of that, of course, the convergence is not as fast as in Table 1), even for z_3 which is close to the singularities of *f*. In the last column the sequence of errors $(R_n(f))_{n\in\mathbb{N}}$ is displayed, where $R_n(f) = f - T_n(fQ)/Q$ and $T_n(fQ)$ denotes the nth Taylor polynomial for *fQ*. Although these last rational approximants seem to be the best, the results in Tables 2 and 3 show that our OPTA provide similar rates of convergence.

Table 2

n	$E_{n,2;1}(z_2)$	$E_{n,2;4}(z_2)$	$E_{n,4;1}(z_2)$	$E_{n,4;4}(z_2)$	$R_n(f)(z_2)$
8	−.1832E-02	−.1792E-02	−.2535E-05	−.3053E-05	−.1755E-02
10	−.6402E-04	−.6253E-04	−.4126E-07	−.4020E-07	−.6139E-04
12	−.1597E-05	−.1559E-05	.1579E-07	−.9049E-07	−.1532E-05
14	−.2216E-07	−.2411E-07	−.7907E-08	−.5816E-06	−.2860E-07
16	.5454E-07	−.6879E-07	−.7393E-08	−.2911E-07	−.4142E-09

Table 3

n	$E_{n,2;1}(z_3)$	$E_{n,3;1}(z_3)$	$E_{n,4;1}(z_3)$	$R_n(f)(z_3)$
8	−.9982E+00	.7123E-01	−.6809E-02	−.9787E+00
10	−.9104E-01	.3890E-02	−.2609E-03	−.8927E-01
12	−.5982E-02	.1712E-03	.1286E-03	−.5866E-02
14	−.2543E-03	.9524E-02	−.5713E-04	−.2900E-03
16	.9182E-03	−.3768E-04	−.1945E-03	−.1117E-04

5. Proofs

For the proof of Proposition 3.1, we need to recall the definition of codimension given in [23, Definition 7.5, p. 22]).

Definition 5.1. Let X be an open subset of $\mathbb{C}^d$. An analytic set $A \subset X$ has codimension s at $a \in A$ (in symbols, $s = codim_a\, A$) if there exists an $s-$dimensional, but no $(s+1)-$dimensional, affine subspace Γ of $\mathbb{C}^d$ such that a is an isolated point of $\Gamma \cap A$. For nonempty A, we define

$$codim\, A = \min_{a \in A} codim_a\, A.$$

Now, we need the following Lemma

Lemma 5.2. *If $p, q \in \mathbb{C}[x_1, \ldots, x_d] \setminus \mathbb{C}$ are relatively prime with $d \geqslant 2$, then the analytic set $p^{-1}(\{0\}) \cap q^{-1}(\{0\})$ has codimension at least 2.*

Proof. The proof is quite simple. Indeed, we can write

$$p(x) = \sum_{i=0}^{l} p_i(x) \text{ and } q(x) = \sum_{i=0}^{m} q_i(x),$$

where for each i, p_i and q_i are homogeneous polynomials of total degree i, with $l, m > 0$ and $p_l q_m \neq 0$. We take $v \in \mathbb{C}^d \setminus \{0\}$ such that $p_l(v)\, q_m(v) \neq 0$. So, it is easy to see that for each $a \in \mathbb{C}^d$ the polynomials $f(t) = p(a+tv)$ and $g(t) = p(a+tv)$ with $t \in \mathbb{C}$ have degree l and m with leading coefficient $p_l(v)$ and $q_m(v)$, respectively. By a linear change of variable, if needed, we can assume that $v = e_1 = (1, 0, \ldots, 0)$.

Let $a \in p^{-1}(\{0\}) \cap q^{-1}(\{0\})$. Applying Proposition 1 given in [11, p. 159], there exist polynomials $A, B \in \mathbb{C}[x_1, \ldots, x_d]$ such that

$$Ap + Bq = \mathrm{Res}(p, q, x_1) \in \mathbb{C}[x_2, \ldots, x_d] \setminus \{0\},$$

where by $\mathrm{Res}(p, q, x_1)$ we denote the *resultant* of p and q with respect to x_1 (for details, see e.g. [11]). In this situation, we only need to know that $\mathrm{Res}(p, q, x_1)$ is a non-zero polynomial which does not depend on x_1.

Take $w \in \mathbb{C}^{d-1} \setminus \{0\}$ such that the polynomial $h(s) = \mathrm{Res}(p, q, x_1)((a_2, \ldots, a_d) + sw)$ does not vanish identically. Let $\Gamma = \left\{x \in \mathbb{C}^d : x = a + s(0, w) + t \cdot e_1, \text{ with } s, t \in \mathbb{C}\right\}$. Thus Γ is a 2−dimensional affine subspace and it is clear that $a \in \Gamma \cap p^{-1}(\{0\}) \cap q^{-1}(\{0\})$. Finally, observe that the set $\Gamma \cap p^{-1}(\{0\}) \cap q^{-1}(\{0\})$ has finite cardinality, since if $x \in \Gamma \cap p^{-1}(\{0\}) \cap q^{-1}(\{0\})$, then $h(s) = 0$ and thus s can only take a finite number of values, $s_1, \ldots, s_k$. Also, for each j, $p\left(a + s_j(0, w) + t \cdot e_1\right) = f(t) = 0$ and so t can only take a finite number of values. □

Now, we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. The result is well known for $d = 1$. Suppose that $d \geqslant 2$. Since $0 \in \mathrm{I}$, we assume that there exists $Q \in \mathrm{I} \setminus \{0\}$ (in other case $\mathrm{I} = (0)$). Without loss of generality, we can suppose that Θ possesses a non-vanishing intersection with each connected component of Ω.

Now, let $Q = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the decomposition of Q in irreducible polynomials of $\mathbb{C}[x_1, \ldots, x_d]$ and consider the family of sets $V_i = \left\{x \in p_i^{-1}\{0\} : \nabla p_i(x) = 0\right\}$ with $i \in \{1, \ldots, k\}$, where ∇f denotes the gradient of f; that is, the vector whose components are the partial derivatives $\frac{\partial f}{\partial x_j}$, with $1 \leqslant j \leqslant d$. Given the open set Ω in $\mathbb{C}^d$, take the new open set $\Omega^* = \Omega \setminus \left(\bigcup_{i=1}^{k}\left(V_i \cup \bigcup_{j=1, j\neq i}^{k} p_i^{-1}\{0\} \cap p_j^{-1}\{0\}\right)\right)$. By applying Lemma 5.2, and taking into account that the finite union of analytic sets of codimension at least 2 also has codimension at least 2 (see e.g. Sections 3.5 and 3.7 in [9]), we obtain that $codim\ \Omega \setminus \Omega^* \geqslant 2$.

On the other hand, if h is a holomorphic extension of fQ on Ω and we denote $\tilde{f} = h/Q$, we have that $\tilde{f} \in \mathcal{O}\left(\Omega^* \setminus \left(\bigcup_{i=1}^{k} p_i^{-1}\{0\}\right)\right)$. Moreover, if we take for each i $(1 \leqslant i \leqslant k)$ the set $S_i = \left\{x \in \Omega^* \cap p_i^{-1}\{0\} : \tilde{f} \notin \mathcal{O}_x\right\}$, where now the notation $g \in \mathcal{O}_x$ means that $g \in \mathcal{O}(\{x\})$, it is easy to see that $\left\{x \in \Omega^* : \tilde{f} \notin \mathcal{O}_x\right\} = \bigcup_{i=1}^{k} S_i$. Thus, if $x \in S_i$ then $\tilde{f} p_i^{\alpha_i} \in \mathcal{O}_x$, and since p_i is irreducible in $\mathcal{O}_x$ (because $\nabla p_i(x) \neq 0$), we have that $\tilde{f} p_i^{\alpha_i} = p_i^m h_1^{\beta_1} \cdots h_s^{\beta_s}$ in terms of the decomposition in irreducible factors of $\mathcal{O}_x$, where $\beta_j \in \mathbb{N}$ for $j \in \{1, \ldots, s\}$ and $m = m(x) \in \mathbb{N}$. Now, for each i with $S_i \neq \emptyset$, consider $x_i \in S_i$ so that $\min\{m(x) : x \in S_i\} = m(x_i) = m_i$ (if $S_i = \emptyset$ we take $m_i = \alpha_i$). We shall see that $\mathrm{I} = (p)$, with $p = \prod_{i=1}^{k} p_i^{\alpha_i - m_i}$.

Indeed, for each i and for any $x \in S_i$, the fact that $\tilde{f} p_i^{\alpha_i} = p_i^m h_1^{\beta_1} \cdots h_s^{\beta_s}$ implies that $\tilde{f} p_i^{\alpha_i - m_i} = p_i^{m - m_i} h_1^{\beta_1} \cdots h_s^{\beta_s} \in \mathcal{O}_x$. Therefore, we have that $\tilde{f} p \in \mathcal{O}\left(\Omega^*\right)$ and, since $codim\ \Omega \backslash \Omega^* \geqslant 2$, by applying the Second Riemann Singularity Theorem (see [23, pp. 23–24]), then $\tilde{f}p \in \mathcal{O}(\Omega)$ and hence, $p \in$ I.

Finally, if $q \in$ I and g, g' are holomorphic extensions of fp and fq on Ω, respectively, then by the Identity Theorem, $q \cdot g = p \cdot g'$ in Ω. So, for any i with $S_i \neq \emptyset$, we have that $g' = \frac{h_1^{\beta_1} \cdots h_s^{\beta_s} q}{p_i^{\alpha_i - m_i}} \in \mathcal{O}_{x_i}$. Thus, $\frac{q}{p_i^{\alpha_i - m_i}} \in \mathcal{O}_{x_i}$ and if we show that this fact implies that $p_i^{\alpha_i - m_i}/q$, for any i, then we conclude that $\prod_{i=1}^{k} p_i^{\alpha_i - m_i} = p/q$ and it settles the proof. It is, however, clear, because if we suppose that $p_i^{\alpha_i - m_i} \nmid q$ for some i, then the decomposition of q in irreducible polynomials will be of the form $q = p_i^s \prod_{j=1}^{k} q_j^{s_j}$, with $s < \alpha_i - m_i$. On the other hand, from $\prod_{j=1}^{k} q_j^{s_j} / p_i^{\alpha_i - m_i - s} \in \mathcal{O}_{x_i}$ we have that $\left(p_i^{\alpha_i - m_i - s}\right)^{-1}(\{0\}) = \left(p_i^{\alpha_i - m_i - s}\right)^{-1}(\{0\}) \cap \left(\prod_{j=1}^{k} q_j^{s_j}\right)^{-1}(\{0\})$ in a neighborhood of x_i. Thus, by applying Lemma 5.2 $\left(p_i^{\alpha_i - m_i - s}\right)^{-1}(\{0\})$ in a neighborhood of x_i will have codimension at least 2 and so $1/p_i^{\alpha_i - m_i - s}$ will admit a holomorphic extension on a neighborhood of x_i by the Second Riemann Singularity Theorem (see [23, pp. 23–24]), which is not possible. $\square$

Now, for the proof of Theorem 3.2 we need the following Lemma given in [9, Lemma 2 pp. 286–287].

Lemma 5.3. *Let f be a function defined in a neighborhood of a set $D' \times \overline{D}_d$, where D' is a domain in $\mathbb{C}^{d-1}$ and $\overline{D}_d$ is a closed, bounded domain in the z_d–plane. Suppose that f is holomorphic in a neighborhood of $D' \times \partial D_d$, and that for each fixed $z' \in D'$ it is holomorphic with respect to z_d in D_d. Then f is holomorphic in $D' \times D_d$.*

Proof of Theorem 3.2. Let $v \in \mathbb{R}^d$ with $v > 0$, $\mu \in [0, 1]$, $\lambda \in \left(\rho_v\left(R, \mathfrak{D}\right), 1\right)$ and denote $r = R\mu^v$ and $\tilde{R} = R\lambda^{-v}$. Since for each k, $r_k = \frac{\tilde{p}_k}{\tilde{q}_k}$, with $\tilde{q}_k$ and $\tilde{p}_k$ as above, for $x \in P\overline{(0, r)}$ we can use the method of proof due to Karlsson and Wallin [22], yielding:

$$
\begin{aligned}
(Qf\tilde{q}_k - Q\tilde{p}_k)(x) = {} & \underbrace{\left(\frac{1}{2\pi i}\right)^d \sum_{\alpha \in (N_k + M)^{\mathrm{c}}} x^\alpha \int_{b_0 P\left(0, \tilde{R}\right)} \frac{(Qf\tilde{q}_k)(y)}{y^{\alpha+1}} dy}_{A(x)} + \\
& \underbrace{\left(\frac{1}{2\pi i}\right)^d \sum_{\alpha \in (N_k + M)} x^\alpha \int_{b_0 P\left(0, \tilde{R}\right)} \frac{(Q(f\tilde{q}_k - \tilde{p}_k))(y)}{y^{\alpha+1}} dy}_{B(x)} .
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
|A(x)| & \leqslant \|Qf\|_{\infty, P\left(0, \tilde{R}\right)} \|\tilde{q}_k\|_{\infty, P\left(0, \tilde{R}\right)} \sum_{\alpha \in \mathbb{N}^d \backslash (N_k + M)} (\mu\lambda)^{<\alpha, v>} \\
& \leqslant const \, \|Qf\|_{\infty, P\left(0, \tilde{R}\right)} \sum_{\alpha \in \mathbb{N}^d \backslash (N_k + M)} (\mu\lambda)^{<\alpha, v>}
\end{aligned}
$$

and then

$$\overline{\lim_{k\to\infty}} \left(\sup\left\{|A(x)| : x \in P\overline{(0,r)}\right\}\right)^{1/\Lambda_v(k)} \leqslant \mu\lambda < 1. \tag{5.1}$$

On the other hand, one also has

$$B(x) = \sum_{\alpha\in E_k} c_\alpha x^\alpha \left(\sum_{\beta\in\{\theta\in M:\alpha+\theta\in(N_k+M)\}} Q_\beta x^\beta \right)$$

with $(f\tilde{q}_k - \tilde{p}_k)(x) = \sum_{\alpha\in\mathbb{N}^d\setminus N_k} c_\alpha x^\alpha$. Hence,

$$\begin{aligned}
|B(x)| &\leqslant const \sum_{\alpha\in E_k} |c_\alpha| R^\alpha \mu^{<\alpha,v>} \\
&\leqslant const \cdot \mu^{\sigma_v(k)} \sum_{\alpha\in E_k} \left| \sum_{\beta\in M} \tilde{q}_{\beta,k} f_{\alpha-\beta} \right| R^\alpha \\
&\leqslant const \cdot u_k \mu^{\sigma_v(k)} \sum_{\alpha\in E_k} \left| \sum_{\beta\in M} q_{\beta,k} f_{\alpha-\beta} \right| R^\alpha \\
&\leqslant const \cdot u_k \mu^{\sigma_v(k)} \delta(k) \sum_{\alpha\in E_k} \left| \sum_{\beta\in M} Q_\beta f_{\alpha-\beta} \right| R^\alpha const^* \\
&\leqslant const \cdot \mu^{\sigma_v(k)} \delta(k) \|Qf\|_{\infty, P\left(0,\tilde{R}\right)} \sum_{\alpha\in E_k} \lambda^{<\alpha,v>},
\end{aligned}$$

where $u_k = |\tilde{q}_k(0)| \left(= 1 = \max_{\beta\in M} \left|\tilde{q}_{\beta,k}\right|, \text{ respect.}\right)$, $\frac{1}{const^*} = |Q(0)| \left(= \max_{\beta\in M} \left|Q_\beta\right|, \text{ respect.}\right)$.

Therefore

$$\overline{\lim_{k\to\infty}} \left(\sup\left\{|B(x)| : x \in P\overline{(0,r)}\right\}\right)^{1/\sigma_v(k)} \leqslant \mu\lambda < 1. \tag{5.2}$$

Thus, from (5.1), (5.2) and since $\Lambda_v(k) \leqslant \sigma_v(k)$, we obtain that $\overline{\lim_{k\to\infty}} \left(\|Qf\tilde{q}_k - Q\tilde{p}_k\|_{\infty,P(0,r)}\right)^{1/\Lambda_v(k)} \leqslant \mu\lambda < 1$, $\mu \in [0,1]$, $\lambda \in \left(\rho_v\left(R,\mathfrak{D}\right),1\right)$. Consequently,

$$\overline{\lim_{k\to\infty}} \left(\|Qf\tilde{q}_k - Q\tilde{p}_k\|_{\infty,P(0,r)}\right)^{1/\Lambda_v(k)} \leqslant \mu\rho_v\left(R,\mathfrak{D}\right) = \rho_v\left(r,\mathfrak{D}\right). \tag{5.3}$$

On the other hand, there exists $P(x) = \sum_{\beta\in M} P_\beta x^\beta$ a polynomial with $1 = \max_{\beta\in M}\left|P_\beta\right|$ and a subsequence $\left(\tilde{q}_{k_j}\right)_{j=0}^\infty$ that converges to P. So, from this and (5.3) we conclude (3.1).

Now, let us see that $fP \in \mathcal{O}(P(0,R))$. Since $fP \in \mathcal{O}\left(P(0,R)\setminus Q^{-1}\{0\}\right)$, it is sufficient to see that for each $x_0 \in P(0,R) \cap Q^{-1}\{0\}$, fP has an analytic extension to a neighborhood of x_0. Thus, using the continuity of such an extension and the fact that $P(0,R)\cap Q^{-1}\{0\} \subset \overline{P(0,R)\setminus Q^{-1}\{0\}}$, the conclusion that fP is holomorphic on $P(0,R)$ easily follows.

To prove it, let $x_0 \in P(0, R) \cap Q^{-1}\{0\}$. We can assume that $x_0 = \left(x_0', x_{0_d}\right) \in \mathbb{C}^{d-1} \times \mathbb{C}$, with $Q\left(x_0', x_{0_d} + \cdot\right)$ not identically equal to zero. Take $\varepsilon, \delta > 0$ so that

$$\left\{x = \left(x', x_{0_d} + t\right) \in \mathbb{C}^d : \left\|x' - x_0'\right\| \leqslant \delta \text{ and } |t| = \varepsilon\right\} = K \subset P(0, R) \setminus Q^{-1}\{0\}$$

and

$$\left\{x = \left(x', x_{0_d} + t\right) \in \mathbb{C}^d : \left\|x' - x_0'\right\| < \delta \text{ and } |t| < 2\varepsilon\right\} = U \subset P(0, R).$$

Thus, $fP \in \mathcal{O}(K)$. For $x' \in \mathbb{C}^{d-1}$ with $\left\|x' - x_0'\right\| < \delta$, we have that $f\left(x', x_{0_d} + \cdot\right)$ is meromorphic in the open disk $\mathbb{D}(0, 2\varepsilon) \subset \mathbb{C}$. Moreover, if it has a pole of order p at $t \in \mathbb{D}(0, 2\varepsilon)$, by (5.3) we have that $P\left(x_0', x_{0_d} + \cdot\right)$ has a zero of order at least p at t, and so $fP\left(x', x_{0_d} + \cdot\right)$ can be extended analytically in $\mathbb{D}(0, 2\varepsilon)$. If we define fP in $U \cap Q^{-1}\{0\}$ making use of this extension, applying Lemma 5.3 with $D' \times D_d = \left\{x = \left(x', x_{0_d} + t\right) \in \mathbb{C}^d : \left\|x' - x_0'\right\| < \delta \text{ and } |t| < \varepsilon\right\}$, we conclude that fP has an analytic extension to a neighborhood of x_0. □

Now, in order to get the proof of Theorem 3.4, we need the following result (Proposition 3.3 in [18]).

Proposition 5.4. *Let* $(N_k)_{k\in\mathbb{N}}, (M_k)_{k\in\mathbb{N}}, (E_k)_{k\in\mathbb{N}}, (\sigma_u(k))_{k\in\mathbb{N}}, (S_u(k))_{k\in\mathbb{N}}, \delta, \sigma_1$ *be as above and* $u \in \left(\mathbb{R}^+\right)^d$ *such that* $\lim_{k\to\infty} \frac{S_u(k)}{\sigma_u(k)} = 1$. *Let* $R > 0$ *be a polyradius and consider a function* f *holomorphic in a neighborhood of the origin. Then, if* $(r_k)_{k=1}^{\infty}$ *is a* σ_1*-geometrically strong (weak) OPTA of* f *for* $\left[(N_k)_{k=1}^{\infty}, (M_k)_{k=1}^{\infty}, R, \delta, \sigma_1\right]$, *we have that for any* $\lambda > 0$, $(r_k)_{k=1}^{\infty}$ *is a* σ_1*-geometrically strong (respec. weak) OPTA of* f *for* $\left[(N_k)_{k=1}^{\infty}, (M_k)_{k=1}^{\infty}, R\lambda^u, \tilde{\delta}, \sigma_1\right]$, *where* $\tilde{\delta} = \left(\tilde{\delta}_k\right)_{k\in\mathbb{N}} \subset [1, \infty)$ *and* $\lim_{k\to\infty} \left(\tilde{\delta}_k\right)^{1/\sigma_1(k)} = 1$.

Proof of Theorem 3.4. From Proposition 5.4 we get that for any $\lambda > 0$, $(r_k)_{k=1}^{\infty}$ is a σ_1-geometrically strong (respec. weak) OPTA of f for $\left[(N_k)_{k=1}^{\infty}, (M_k)_{k=1}^{\infty}, R\lambda^u, \tilde{\delta}, \sigma_1\right]$. So, if we apply the result of Theorem 3.2 with $\lambda > 0$ so that $P\overline{\left(0, R\lambda^u\right)} \subset \mathfrak{D}$, we conclude (3.2) and the fact that Q/P follows taking into account that $P \in \mathrm{I}\left(f, P\left(0, \tilde{R}\right)\right)$. □.

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